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On deformations of a rotating disk of electrostrictive material characterized by a viscoelasticity of Voigt type

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The object of the present paper is to investigate the nature of the deformations of a rotating disk of electrostrictive material characterized by a viscoelasticity of Voigt type. The method of Laplace-transforms has been found effective in making the problem amenable to solution.

1. Introduction

The phenomenon of electrostriction; as is well-known, is the outcome of the interaction of two fields *i.e.* mechanical and electrical. The problems on dielectrics, subjected to suitable electric fields, can be extended to electro-strictive problems when appropriate elastic fields are accommodated in the former type of problems. The electrostrictive problems have been investigated by several authors in recent years. The recent paper of Sinha (1969) may be mentioned in this context. The present paper sets out to consider the nature of the deformations in a rotating disk of electrostrictive material having the property of visco-elasticity of Voigt type. The solution of this problem has been facilitated by the use of Laplace transforms. Finally the displacement at a point on the boundary has been evaluated completely. The nature of the expression for the displacement brings out the fact it is composed of two parts, one which is time-dependent and the other which remains steady.

2. Problem and basic equations

Let us consider a thin rotating electrostrictive disk of radius a and of length $2l$, having the property of viscosity of Voigt type. The disk is rotating with the given angular velocity Ω. Our purpose is to find out the deformations due to the interaction of two fields, i. e. mechanical and electrical. To work out the expressions for deformations, we have to fall back upon the equations of equilibrium, expressed in terms of displacements. These equations of equilibrium can be solved only when the relevant constitutive relations are made use of. The constitutive relations of an electrostrictive material as in Knops (1963) have been modified for the viscoelastic problem and they are given by, when referred to coordinates (r, θ, z),

$$\sigma_{rr}=\left(\lambda+\lambda'\frac{\partial}{\partial t}\right)(S_{rr}+S_{\theta\theta}+S_{zz})+2\left(\mu+\mu'\frac{\partial}{\partial t}\right)S_{rr}$$
$$+(a_1+b_1)E_r^2+a_1E_\theta^2+a_1E_z^2$$

$$\sigma_{\theta\theta}=\left(\lambda+\lambda'\frac{\partial}{\partial t}\right)(S_{rr}+S_{\theta\theta}+S_{zz})+2\left(\mu+\mu'\frac{\partial}{\partial t}\right)S_{\theta\theta}$$
$$+a_1E_r^2+(a_1+b_1)E_\theta^2+a_1E_z^2$$

$$\sigma_{zz}=\left(\lambda+\lambda'\frac{\partial}{\partial t}\right)(S_{rr}+S_{\theta\theta}+S_{zz})+2\left(\mu+\mu'\frac{\partial}{\partial t}\right)S_{zz}$$
$$+a_1E_r^2+a_1E_\theta^2+(a_1+b_1)E_z^2$$

$$\sigma_{\theta z}=2\left(\mu+\mu'\frac{\partial}{\partial t}\right)S_{\theta z}+b_1E_\theta E_z.$$

$$\sigma_{rz}=2\left(\mu+\mu'\frac{\partial}{\partial t}\right)S_{rz}+b_1E_rE_z.$$

$$\sigma_{r\theta}=2\left(\mu+\mu'\frac{\partial}{\partial t}\right)S_{r\theta}+b_1E_rE_\theta.$$

where σ_{rr}, etc. are stress components, S_{rr}, etc. are strain components, E_r, E_θ, E_z are components of electric intensity, λ, λ', μ, μ' are the material parameters while a_1, b_1 are electrostrictive constants.

Referred to coordinates (r, θ, z,) the stress equation of equilibrium for a rotating disk, as suggested by Timoshenko & Goodier (1934) is

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}+\sigma\Omega^2r=0 \quad \text{.. (1)}$$

where ρ is the mass per unit volume of the material of the disk.

The electric field is governed by the equations

$$rot\vec{E}=0. \quad \text{.. (2)}$$

$$div\vec{D}=0. \quad \text{...(3)}$$

$$\vec{D}=K\vec{E} \quad \text{...(4)}$$

where $\vec{D}$ and $\vec{E}$ are the electric displacement vector and the electric intensity vector respectively, K is the specific inductive capacity.

The solution of the equation of equilibrium is subject to the usual Foundary conditions, the most important of which is

$$\int_{-l}^{+l} \sigma_{rr} . dz = 0 \quad on \ r = a$$

3. Solution of the problems :

The corresponding problem for the purely elastic case has been solved in the treatise of Love (1927) and the present solution broadly conforms to this procedure. Therefore, we consider the problem as one of plane stress in order to obtain an approximate solution of the problem so that

$$\sigma_{zz} = 0. \qquad \sigma_{rz} = 0.$$

throughout the disk.

Let u and w be the respective radial and longitudinal displacements, then

$$\left.\begin{aligned} S_{rr} &= \frac{\partial u}{\partial r}\,,\ S_{\theta\theta} = \frac{u}{r}\,,\ S_{zz} = \frac{\partial w}{\partial z} \\ S_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\,, S_{r\theta} = 0\,,\ S_{\theta z} = 0\,. \end{aligned}\right\} \quad ..(5)$$

when u and w are functions of both r and z.

Let V be the electric potential. Then from (2) $\vec{E} = -\operatorname{grad} V$ ...(6)

From (3) and (4)

$$\operatorname{div}\ (K \operatorname{grad} V) = 0\,. \qquad ...(7)$$

Since K is a constant and V is a function of Z, we get from (7)

$$\frac{d^2 V}{dz^2} = 0.$$

The solution of this equation is given by

$$V = Az + B$$

where A and B are constants to be determined from the potentials at the ends of the disk. Therefore $B = \frac{V_l - V_{-l}}{2}$ and $A = \frac{V_l - V_{-l}}{2} = Eo$ (say)

Thus, the components of the electric field are given by

$$\left.\begin{aligned} E_r &= -\frac{\partial V}{\partial r} = 0 \\ E_\theta &= -\frac{\partial V}{r\partial \theta} = 0 \\ E_z &= -\frac{\partial V}{\partial z} = -A = -E_0. \end{aligned}\right\} \quad ...(8)$$

Considering the equations (5) and (8) we have

$$\sigma_{rr} = \left(\lambda+\lambda'\frac{\partial}{\partial t}+2\mu+2\mu'\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial r}+\left(\lambda+\lambda'\frac{\partial}{\partial t}\right)\frac{u}{r}$$

$$+\left(\lambda+\lambda'\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial z}+a_1 E_0^2 \qquad \dots (9)$$

$$\sigma_{\theta\theta} = \left(\lambda+\lambda'\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial r}+\left(\lambda+\lambda'\frac{\partial}{\partial t}+2\mu+2\mu'\frac{\partial}{\partial t}\right)\frac{u}{r}$$

$$+\left(\lambda+\lambda'\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial z}+a_1 E_0^2 \qquad \dots (10)$$

$$0 = \sigma_{zz} = \left(\lambda+\lambda'\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial r}+\left(\lambda+\lambda'\frac{\partial}{\partial t}\right)\frac{u}{r}$$

$$+\left(\lambda+\lambda'\frac{\partial}{\partial t}+2\mu+2\mu'\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial z}+(a_1+b_1)E_0^2 \qquad \dots (11)$$

$$\sigma_{\theta z}=0 \qquad \dots (12)$$

$$0 = \sigma_{rz} = \left(2\mu+2\mu'\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right) \qquad \dots (13)$$

$$\sigma_{r\theta} = 0 \qquad \dots (14)$$

Let $\bar{u}$ and $\bar{w}$ denote Laplace transforms of u and w respectively.

Thus

$$\bar{u} = \int_0^\infty u \exp(-pt)\, dt.$$

$$\bar{w} = \int_0^\infty w \exp(-pt)\, dt.$$

Let the system be stressed by sudden application of forces so that initially all the unknown quantities are zero.

Then $\int_0^\infty \left(\lambda+\lambda'\frac{\partial}{\partial t}\right) u \exp(-pt)\, dt = \lambda_1\bar{u}$

where $\lambda_1 = \lambda+\lambda' p$

Similarly, $\int_0^\infty \left(\mu+\mu'\frac{\partial}{\partial t}\right) u \exp(-pt)\, dt = \mu_1\bar{u}$

where $\mu_1 = \mu+\mu' p$

From (13),

$$\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial r}$$

Therefore

$$\frac{\partial \bar{u}}{\partial z} = -\frac{\partial \bar{w}}{\partial r}. \qquad \ldots (15)$$

From (11), we have

$$\frac{\partial \bar{w}}{\partial z} = -\frac{\lambda_1}{(\lambda_1+2\mu_1)}\left(\frac{\partial u}{\partial r}+\frac{\bar{u}}{r}\right)-\frac{(a_1+b_1)E_o{}^2}{(\lambda_1+2\mu_1)p} \qquad \ldots (16)$$

Using (9), (10) and $\lambda_1 = \lambda + \lambda' p$, $\mu_1 = \mu + \mu' p$ in (1) we have

$$(\lambda_1+2\mu_1)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial \bar{u}}{\partial r}-\frac{\bar{u}}{r^2}\right)+\lambda_1\frac{\partial^2 \bar{w}}{\partial z \partial r} = -\frac{\rho \Omega^2 r}{p}.$$

From (16)

$$\frac{\partial^2 \bar{w}}{\partial r \partial z} = -\frac{\lambda_1}{(\lambda_1+2\mu_1)}\frac{\partial}{\partial r}\left(\frac{\partial \bar{u}}{\partial r}+\frac{\bar{u}}{r}\right) \qquad \ldots (17)$$

Therefore,

$$(\lambda_1+2\mu_1)\left(\frac{\partial^2 \bar{u}}{\partial r^2}+\frac{1}{r}\frac{\partial \bar{u}}{\partial r}-\frac{\bar{u}}{r^2}\right)+\frac{\lambda_1{}^2}{(\lambda_1+2\mu^1}\frac{\partial}{\partial r}\left(\frac{\partial \bar{u}}{\partial r}+\frac{\bar{u}}{r}\right) = -\frac{\rho \Omega^2 r}{p}.$$

$$\text{or } 4\mu_1(\lambda_1+\mu_1)\frac{\partial}{\partial r}\left(\frac{\partial \bar{u}}{\partial r}+\frac{\bar{u}}{r}\right) = -\frac{(\lambda_1+2\mu_1)}{p}(\rho \Omega^2 r). \qquad \ldots (18)$$

Again from (15), (17) and (18)

$$\frac{\partial^2 \bar{u}}{\partial z^2} = -\alpha r. \qquad \ldots (19)$$

where

$$\alpha = \frac{\lambda_1 \rho \Omega^2}{4\mu_1(\lambda_1+\mu_1)p}.$$

The solution of (19) is

$$\bar{u} = -\tfrac{1}{2}\alpha r z^2 + z f_1(r) + f_2(r) \qquad \ldots (20)$$

where $f_1(r)$ and $f_2(r)$ are functions of r to be determined. To find $f_1(r)$ and $f_2(r)$ we substitute this value of $\bar{u}$ in (18). Thus we have

$$\left[f''_1(r)+\frac{f'_1(r)}{r}-\frac{f_1(r)}{r^2}\right]z+\left[f''_2(r)+\frac{f'_2(r)}{r}-\frac{f_2(r)}{r^2}+\delta r\right]=0 \quad ...(21)$$

where $\delta = \dfrac{(\lambda_1+2\mu_1)\rho\Omega^2}{4\mu_1(\lambda_1+\mu_1)p}$

If this equation (21) is to hold for all values of z, we must have

$$f_1''(r)+\frac{f_1'(r)}{r}-\frac{f_1(r)}{r^2}=0$$

$$f_2''(r)+\frac{f_2'(r)}{r}-\frac{f_2(r)}{r^2}+\delta r=0$$

Thus $f_1(r) = A_1 r + \dfrac{A_2}{r}$ ... (22)

$$f_2(r) = A_3 r + \frac{A_4}{r} - \frac{1}{8}\delta r^3 \quad .. (23)$$

where A_1, A_2, A_3, A_4, are arbitrary constants of integration to be determined from the boundary conditions.

We now impose the condition (Love 1927) that $u=0$ at $r=0$, $z=0$, *i.e.* $\bar{u}=0$ at $r=0$, $z=0$,. With these values of $f_1(r)$ and $f_2(r)$ we get from (20),

$$\bar{u} = -\frac{1}{2}\alpha rz^2 + A_1 rz + \frac{A_2 z}{r} + A_3 r + \frac{A_4}{r} - \frac{1}{8}\delta r^3$$

Putting $\bar{u}=o$ at $r=o$, $z=o$, we have $A_2 = A_4 = o$. Imposing now the condition as in (4) that $\dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial r} = 0$ at $r=0$, $z=0$ which means from (15), $\dfrac{\partial \bar{u}}{\partial z} = 0$

Applying this condition to the expression for $\bar{u}$, we have $A_1 = 0$.

$$\therefore \quad \bar{u} = -\frac{1}{2}\alpha rz^2 + A_3 r - \frac{1}{8}\delta r^3$$

To evaluate the constant A_3, we shall use the boundary codition

$$\int_{-l}^{+l} \sigma_{rr}\,dz = 0 \quad \text{on } r=a.$$

That is, $\displaystyle\int_{-l}^{+l} \bar{\sigma}_{rr}\,dz = 0$ on $r=a$.

Now from (9)

$$\overline{\sigma}_{rr} = -\frac{(3\lambda_1+2\mu_1)\mu_1}{(\lambda_1+2\mu_1)} \alpha z^2 + \frac{2(3\lambda_1+2\mu_1)\mu_1}{(\lambda_1+2\mu_1)} A_3 - \frac{(7\lambda_1+6\mu_1)\mu_1}{4(\lambda_1+2\mu_1)} \delta r^2 - \frac{(\lambda_1 b_1-2\mu_1 a_1)E_0^2}{p(\lambda_1+2\mu_1)}$$

Therefore $\int_{-l}^{+l} \overline{\sigma}_{rr}\, dz=0$ on $r=a$ gives on simplification

$$A_3 = \frac{\alpha l^2}{6} + \frac{7\lambda_1+6\mu_1}{8(3\lambda_1+2\mu_1)} \delta a^2 + \frac{(b_1\lambda_1-2_1\mu a_1)E_0^2}{2\mu_1(3\lambda_1+2\mu_1)p} \qquad \dots (25)$$

Hence $\bar{u}$ is completely known from (24) and (25). By taking its inverse transform, we can determine u. In particular, to obtain the value of u, at the point ($r=a$, $z=l$), we have from (24) and (25),

$$u\Big]_{r=a,\, z=l} = -\frac{a\alpha l^2}{3} + \frac{(7\lambda_1+6\mu_1)a^3\delta}{8(3\lambda_1+2\mu_1)} + \frac{(b_1\lambda_1-2\mu_1 a_1)aE_0^2}{2\mu_1(3\lambda_1+2\mu_1)p} - \frac{1}{8}\delta a^3 . \qquad \dots (26)$$

Now inverse transform of α is

$$\frac{\lambda' \rho\Omega^2}{4\mu'(\lambda'+\mu')}\left[\frac{L}{MN} + \frac{L-M}{M(M-N)} e^{-Mt} + \frac{L-N}{N(N-M)} e^{-Nt}\right] \qquad \dots (27)$$

Inverse transform of δ is

$$\frac{(\lambda'+2\mu')\rho\Omega^2}{4\mu'(\lambda'+\mu')}\left[\frac{L_1}{MN} + \frac{L_1-M}{M(M-N)} e^{-Mt} + \frac{L_1-N}{N(N-M)} e^{-Nt}\right] \qquad \dots (28)$$

Inverse transform of $\dfrac{(7\lambda_1+6\mu_1)\delta}{(3\lambda_1+2\mu_1)}$ is

$$\frac{(7\lambda'+6\mu')(\lambda'+2\mu')\rho\Omega^2}{4\mu'(\lambda'+\mu')\, 3\lambda'+2\mu'}\left[\frac{L_1 L_2}{MNW} H(t) + \frac{L_1 L_2 - M(L_1+L_2)+M}{M(M-N)(W-M)} e^{-Mt}\right.$$
$$\left. + \frac{L_1 L_2-(L_1+L_2)N+N}{N(W-N)(N-M)} e^{-nt} + \frac{L_1 L_2-(L_1+L_2)W+W}{W(M-W)(W-N)} e^{-Wt}\right] \qquad \dots (29)$$

Inverse transform of $\dfrac{(b_1 \lambda_1-2\mu_1 a_1)}{2\mu_1(3\lambda_1+2\mu_1)p}$ is

$$\frac{(b_1\lambda'-2\mu' a_1)}{2\mu'(2\lambda'+2\mu')}\left[\frac{K}{MW} + \frac{K-M}{M(M-W} e^{-Mt} + \frac{K-W}{W(W-M)} e^{-Wt}\right] \qquad \dots (30)$$

where

$$L = \frac{\lambda}{\lambda'}, \quad M = \frac{\mu}{\mu'}, \quad N = \frac{\lambda+\mu}{\lambda'+\mu'}, \quad L_1 = \frac{\lambda+2\mu}{\lambda'+2\mu'},$$

$$L_2 = \frac{7\lambda+6\mu}{7\lambda'+6\mu'}, \quad K = \frac{b_1\lambda-2\mu a_1}{b_1\lambda'-2\mu' a_1}, \quad W = \frac{3\lambda+2\mu}{3\lambda'+2\mu'},$$

$$H(t) \left.\begin{array}{l} = 1, \quad t>0 \\ = 0, \quad t<0 \end{array}\right\}$$

Now considering all the results (27), (28), (29), and (30), we have from (26), at ($r=a$, $z=l$) in the form given by

$$\begin{aligned}
u\Big]_{r=a,\, z=l} = & \Bigg[-\frac{\lambda'\rho\Omega^2 a l^2 L}{12\mu'(\lambda'+\mu')MN} - \frac{(\lambda'+2\mu')\rho\Omega^2 a^3 L_1}{32\mu'(\lambda'+\mu')MN} \\
& + \frac{(b_1\lambda'-2\mu' a_1)aE_0^2 K}{2\mu'(3\lambda'\div 2\mu')MW}\Bigg] + \\
& + \Bigg[-\frac{\lambda'\rho\Omega^2 a l^2(L-M)}{12\mu'(\lambda'+\mu')M(M-N)} - \frac{(\lambda'+2\mu')\rho\Omega^2 a^3(L_1-M)}{32\mu'(\lambda'+\mu')M(M-N)} + \\
& + \frac{(7\lambda'+6\mu')(\lambda'+2\mu')\rho\Omega^2 a^3\{L_1L_2-M(L_1+L_2)+M\}}{32\mu'(\lambda'+\mu')+3\mu'+2\mu'(M(M-N)(W-M)} + \\
& + \frac{(b_1\lambda'-2\mu' a_1)aE_0^2(K-M)}{2\mu'(3\lambda'+2\mu')M(M-W)}\Bigg] e^{-Mt} + \\
& + \Bigg[-\frac{\lambda'\rho\Omega^2 a l^2(L-N)}{12\mu'(\lambda'+\mu')N(N-M)} - \frac{(\lambda'+\mu')\rho\Omega^2 a^3(L_1-N)}{32\mu'(\lambda'+\mu')N(N-M)} + \\
& + \frac{(7\lambda'+6\mu')(\lambda'+2\mu')\rho\Omega^2 a^3\{L_1L_2-(L_1+L_2)N+N\}}{32\mu'(\lambda'+\mu')(3\lambda'+2\mu')N(W-N)(N-M)}\Bigg] e^{-Nt} \\
& + \Bigg[\frac{(7\lambda'+6\mu')(\lambda'+2\mu')\rho\Omega^2 a^3\{L_1L_2-(L_1+L_2)W+W\}}{32\mu'(\lambda'+\mu')(3\lambda'+2\mu')W(M-W)(W-N)} \\
& + \frac{(b_1\lambda'-2\mu' a_1)aE_0^2(K-W)}{2\mu'(3\lambda'+2\mu')W(W-M)}\Bigg] e^{-Wt} + \\
& + \frac{(7\lambda'+6\mu')(\lambda'+2\mu')\rho\Omega^2 L_1 L_2}{4\mu'(\lambda'+\mu')(3\lambda'+2\mu')MNW} H(t) .
\end{aligned}$$

Thus the expression for the displacement shows that it consists of two parts, namely, the time-independent part and the time-dependent part. The first part, being time-independent, remains unaffected. with the increase of time and therefore represents the steady state solution.

Having determined $\bar{u}$ we can similarly determine the longitudinal deformation w fom (16) by a simple integration, the constant of integration being zero when we impose the condition $r=0$ when $z=0$

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